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LETTER TO THE EDITOR

New method of structural functions for analysing fractal scaling properties of natural processes

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Abstract. A method to reveal and estimate the fractal scaling properties of positive and negative increments in one-valued functions that describe natural processes is proposed. Structural functions, which were introduced by Kolmogorov in 1941 for analysing the scaling properties of small-scale turbulence, provide the basis of the method. Examples are given to illustrate the application of the proposed method for analysing the trace of one of the coordinates of Brownian motion, simulated asymmetric wave forms, internal waves, sand waves in one-directional streams and river turbulence.

Various methods are now applied for analysing the fractal scaling properties of one-valued functions which describe natural processes. Among the most well known are Hurst's method, Richardson's method, the box-counting method as well as that of spectral densities [1–3]. However, these methods are characterized by certain drawbacks and restrictions thus stimulating the elaboration of new approaches [4–6]. All the above mentioned methods are aimed at revealing self-similarity or self affinity properties and at determining their quantitative characteristics, namely, fractal dimensions and Hurst's exponent, without paying attention to the possible asymmetry of processes relative to their peak values. Thus, for instance, the formation of the relief of northern and southern slopes on the earth's surface may have different peculiarities under certain circumstances. On a profile of the earth's surface this would be expressed in the asymmetry of the shape of its rough projections. Various wavy phenomena, having asymmetrical shape as a result of nonlinear or other effects, may serve as another example. The above-mentioned methods for analysis of such phenomena do not give information about their possible asymmetry.

In our opinion, the following approach also allows, along with traditional scaling exponents, determination of the properties of positive and negative increments in functions that describe natural processes.

We shall take the method of structural functions introduced by Kolomogorov in 1941 for analysing the structure of small-scale turbulence [7] to serve as the basis of our approach. For one-valued function $y(x)$ the structural function is determined by the relationship

$$\overline{\Delta y^2}(\Delta x) = \overline{[y(x+\Delta x) - y(x)]^2} \tag{1}$$

where the line on top points to the averaging operation in respect of the realizations ensemble of $y(x)$ or, when adopting the ergodicity hypothesis, in respect to realization length. When scaling exists in the structure of function $y(x)$, relationship (1) acquires the power form

$$\overline{\Delta y^2}(\Delta x) \sim \Delta x^{2H} \qquad (2)$$

where H is Hurst's exponent [1, 2]. It is connected with the power spectrum $S_y(f) \sim f^{-\beta}$ of function $y(x)$ by the relationship $\beta = 2H + 1$ [1–3]. The structural function $\overline{\Delta y^2}(\Delta x)$ is quite an effective instrument for revealing and estimating fractal scaling properties and has enjoyed rather wide use lately in solving various physical problems [3, 8, 9]. Indeed, it has an obvious advantage over Hurst's, Richardson's and box-counting methods since it does not require preliminary information concerning the structure of the investigated function $y(x)$ or of any of its preliminary transformations (e.g. compression of the $y(x)$ graph in Richardson's and box-counting methods for determining H [2, 3]).

The essence of our innovation is in the separate consideration of positive Δy_+ and negative Δy_- increments in function $y(x)$. In other words, in addition to (1) we suggest the following functions:

$$\overline{\Delta y_+^2}(\Delta x) = \overline{[y(x+\Delta x) - y(x)]^2} \qquad \Delta y(\Delta x) > 0 \qquad (3)$$

$$\overline{\Delta y_-^2}(\Delta x) = \overline{[y(x+\Delta x) - y(x)]^2} \qquad \Delta y(\Delta x) < 0 \qquad (4)$$

which characterize the structure of positive (3) and negative (4) increments in the function $y(x)$. It is obvious that, generally, functions $\overline{\Delta y_+^2}$ and $\overline{\Delta y_-^2}$ can be presented as follows:

$$\overline{\Delta y_+^2}(\Delta x) = \overline{\Delta y_+}^2(\Delta x) + \overline{\Delta y_+'^2}(\Delta x) \qquad (5)$$

$$\overline{\Delta y_-^2}(\Delta x) = \overline{\Delta y_-}^2(\Delta x) + \overline{\Delta y_-'^2}(\Delta x) \qquad (6)$$

where $\overline{\Delta y_+}$ and $\overline{\Delta y_-}$ are the average positive and negative increments characterizing the positive and negative slopes of $y(x)$, $\overline{\Delta y_+'^2}$ and $\overline{\Delta y_-'^2}$ are their variances. Thus relationships (3)–(6) introduce new characteristics for consideration which reflect properties of the investigated processes that are inaccessible with conventional methods. Obviously, for processes displaying symmetrical peaks, the characteristics of positive and negative increments will coincide. For this case the relationship $\overline{\Delta y_+^2}(\Delta x) = \overline{\Delta y_-^2}(\Delta x) = \overline{\Delta y^2}(\Delta x)$ will occur. This is illustrated by figure 1 which shows the characteristics we have introduced, for a trace of Brownian motion. However, if we have non-symmetry in wave shape or in function $y(x)$ projections which are different at various scales, we shall obtain different behaviour of the proposed new characteristics $(\overline{\Delta y_+^2}, \overline{\Delta y_-^2}, \overline{\Delta y_+}^2, \overline{\Delta y_-}^2, \overline{\Delta y_+'^2}, \overline{\Delta y_-'^2})$. It is not excluded that their scaling exponents will be different. In this connection it should be noted that our approach is related, to a certain degree, with multifractal methodology [2]: we consider the subsets for negative $(S\Delta x_-)$ and positive $(S\Delta x_-)$ increments which give in their sum the set of all increments, $S\Delta x = S\Delta x_+ U S\Delta x_-$. In some cases we may expect some differences between prefactors and exponents in scaling relationships for the new structural functions (5) and (6). The new characteristics are presented in figure 2

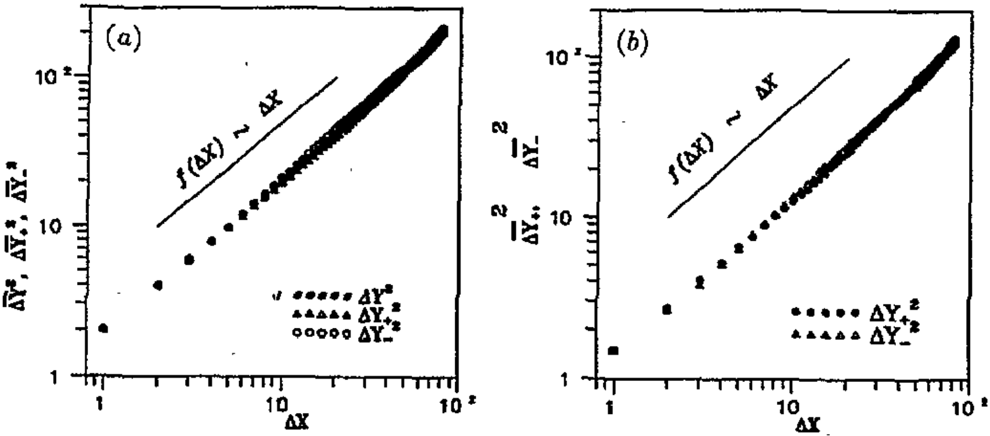

Figure 1. Graphs of functions $\overline{\Delta y^2}$, $\overline{\Delta y_+^2}$, $\overline{\Delta y_-^2}$ (*a*) and $\overline{\Delta y_+}^2$, $\overline{\Delta y_-}^2$ (*b*) for one of the Brownian motion coordinates (computer simulation).

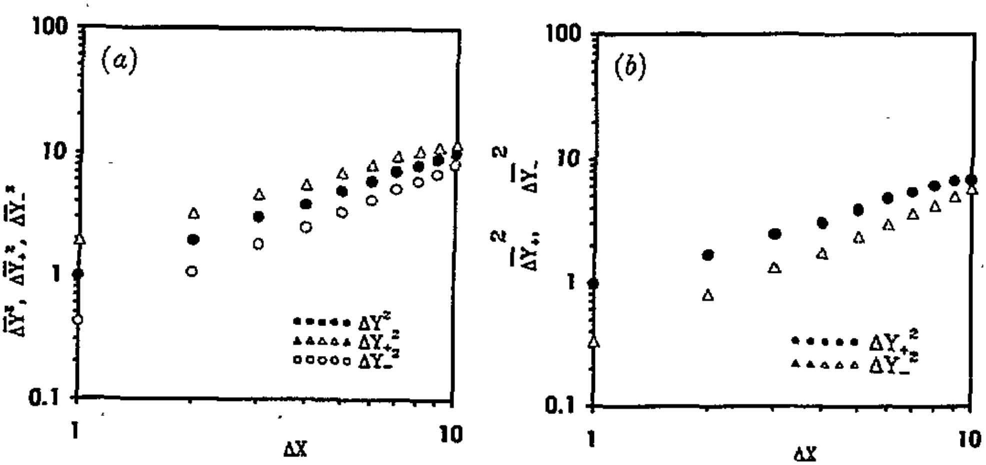

Figure 2. Graphs of functions $\overline{\Delta y^2}$, $\overline{\Delta y_+^2}$, $\overline{\Delta y_-^2}$ (*a*) and $\overline{\Delta y_+}^2$, $\overline{\Delta y_-}^2$ (*b*) for summed, exponentially distributed, random values, variance 1, ranging from -1 to infinity (computer simulation).

for a function with asymmetric peaks, being generated by the summation of exponentially distributed random numbers with variance 1, ranging from -1 to infinity.

In connection with the discussion of our approach it should be mentioned that the third-order structural function $\overline{\Delta y^3}(\Delta x)$ is sometimes applied to evaluate the asymmetry of waves or projections of function $y(x)$ [7, 8]. However, we regard our approach as being more convenient: it simultaneously allows estimation of both the usual scaling exponents and the degree of asymmetry. Besides, it has better physical clarity and is easier for physical interpretation.

Hereafter we shall apply the described approach for analysing the structure of internal waves (A), sand waves in a one-direction stream (B) and river turbulence (C). We shall not subject the revealed effects to a detailed analysis but present the results to show the possibilities of the proposed method. The data used for calculations were obtained during laboratory and field experiments.

(A). To serve as initial data, we use measurements of water temperature averaged by depth, $T(t)$, which were carried out in the Black Sea, 2 km off shore. Joint analysis of vertical profiles of temperature and salinity shows the function $T(t)$ is linearly connected with the fluctuations of the depth of the internal boundary dividing warm and cold water. To a certain extent this provides us with the possibility of obtaining information about the structure of internal waves on the basis of time fluctuation of depth averaged temperature. The utilization of the described approach applied to $T(t)$ is given in figure 3. Joint analysis of the graphs in figure 3 allows us to draw the following conclusions. The investigated range of time scales of $T(t)$ can be divided into three sub-ranges (figure 3).

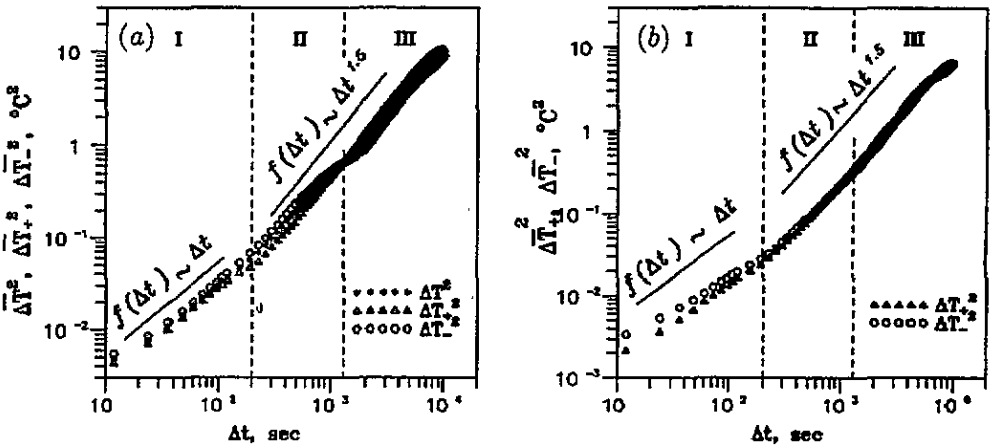

Figure 3. Graphs of functions $\overline{\Delta T^2}$, $\overline{\Delta T_+^2}$, $\overline{\Delta T_-^2}$ (a) and $\overline{\Delta T_+}^2$, $\overline{\Delta T_-}^2$ (b) for temperature fluctuations $T(t)$ (measurements in the Black Sea).

For the first sub-range $\overline{\Delta T_-^2} > \overline{\Delta T_+^2}$, $\overline{\Delta T_-}^2 > \overline{\Delta T_+}^2$ and $\overline{\Delta T_-'^2} < \overline{\Delta T_+'^2}$ are characteristic. The forward face of small-scale waves is, on average, steeper than the back face ($\overline{\Delta T_-}^2 > \overline{\Delta T_+}^2$). The fluctuations in slope of the internal wave surface in this sub-range is slightly smaller for the front face of waves than for the back one ($\overline{\Delta T_-'^2} < \overline{\Delta T_+'^2}$).

In going from the first sub-range to the second one, the situation changes and we have $\overline{\Delta T_-^2} > \overline{\Delta T_+^2}$, $\overline{\Delta T_-}^2 \approx \overline{\Delta T_+}^2$ and $\overline{\Delta T_-'^2} > \overline{\Delta T_+'^2}$, i.e., on average the wave shape is symmetrical but the fluctuations in slope of the internal wave surface is bigger for the forward face of the waves.

For the third sub-range the relationship $\overline{\Delta T_-}^2 \approx \overline{\Delta T_+}^2$ holds, however, the relation between $\overline{\Delta T_-'^2}$ and $\overline{\Delta T_+'^2}$ and between $\overline{\Delta T_-^2}$ and $\overline{\Delta T_+^2}$ reverses: $\overline{\Delta T_-'^2} < \overline{\Delta T_+'^2}$ and $\overline{\Delta T_-^2} < \overline{\Delta T_+^2}$. It should be added that scaling exponents H for functions $\overline{\Delta T^2}$, $\overline{\Delta T_+^2}$, $\overline{\Delta T_-^2}$, $\overline{\Delta T_+}^2$, $\overline{\Delta T_-}^2$, $\overline{\Delta T_+'^2}$, $\overline{\Delta T_-'^2}$ are close to each other. For the first sub-range they are approximately equal to 0.5, for the second and third ones they are close to 0.75 (figure 3). Thus in both cases the geometry of internal waves is characterized by self-affine structure ($H < 1$). The given analysis points to the possible existence of three mechanisms for generation and development of the investigated internal waves, each associated with a particular time interval: (1) $\Delta t \leqslant 200$ sec, (2) 200 sec $\leqslant \Delta t \leqslant 1300$ sec and (3) $\Delta t \geqslant 1300$ sec. These properties have been revealed on the basis of the proposed approach and would have been inaccessible with traditional fractal scaling analysis methods.

(B). Here we shall analyse the scale properties of bed elevation $Z(x)$ and $Z(t)$, representing longitudinal and time profiles of sand waves in one direction along an alluvial stream. The experiments were conducted in flumes at the State Hydrological Institute (St. Petersburg) and at the Odessa Hydrometeorological Institute [10].

The characteristic graphs of functions (1), (3)–(6) for longitudinal profiles of sandy stream bed $Z(x)$ are shown in figure 4. For small scales the inequalities $\overline{\Delta Z_+^2} > \overline{\Delta Z_-^2}$, $\overline{\Delta Z_+}^2 > \overline{\Delta Z_-}^2$ and $\overline{\Delta Z_+^{2}} > \overline{\Delta Z_-^{2}}$ are evident, thus providing evidence of the asymmetrical shapes of sand waves: the back face of sand bed waves is more gentle than the forward one. Further, notice is drawn to the difference in scaling exponents for positive ($H_+ \approx 1.0$) and negative ($H_- \approx 0.8$) increments (figure 4). It is well known that the forward face of sand waves, characterized in our approach by positive increments, has a constant slope (close to the angle of internal friction of sand in water [10]) which stipulates its independence of sand wave scale. From this follows the self-similarity for positive increments of $Z(x)$. This is expressed by the relationships $\overline{\Delta Z_+^2} \sim \overline{\Delta Z_+}^2 \sim \Delta x^{2H+}$ with $H_+ = 1$. At the same time, one can observe the self-affinity of negative increments with $H_- \approx 0.8 < 1.0$ which testifies to the decrease of the inclination angle of sand waves back face when the scale of waves increases.

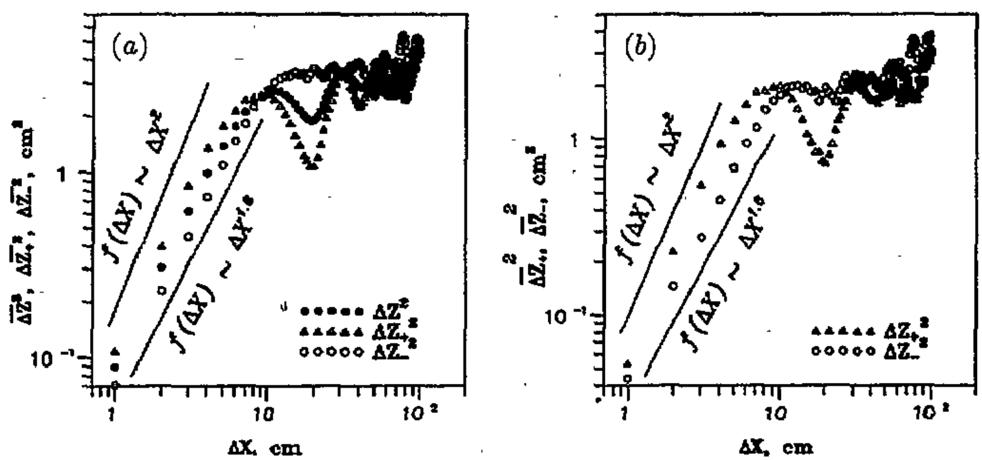

Figure 4. Characteristic graphs of functions $\overline{\Delta Z^2}$, $\overline{\Delta Z_+^2}$, $\overline{\Delta Z_-^2}$ (*a*) and $\overline{\Delta Z_+}^2$, $\overline{\Delta Z_-}^2$ (*b*) for longitudinal profiles $Z(x)$ of bottom sand waves (laboratory experiments).

Analysis of functions (1) and (3)–(6) for time changes of bottom elevation $Z(t)$ in the case of sand waves passing a point (figure 5) also gives evidence of asymmetric fluctuations of the bottom elevation. The waves forward face, characterized by negative increments in function $Z(t)$, is steeper compared to the back face ($\overline{\Delta Z_+^2} > \overline{\Delta Z_-^2}$, $\overline{\Delta Z_+}^2 > \overline{\Delta Z_-}^2$). Besides, in the case of small Δt one can also observe distinct scaling behaviour of functions $\overline{\Delta Z_+^2}$, $\overline{\Delta Z_+}^2$ and $\overline{\Delta Z_-^2}$, $\overline{\Delta Z_-}^2$ with various scaling exponents: $H_+ \approx 1.0$ and $H_- \approx 0.65$ (figure 5).

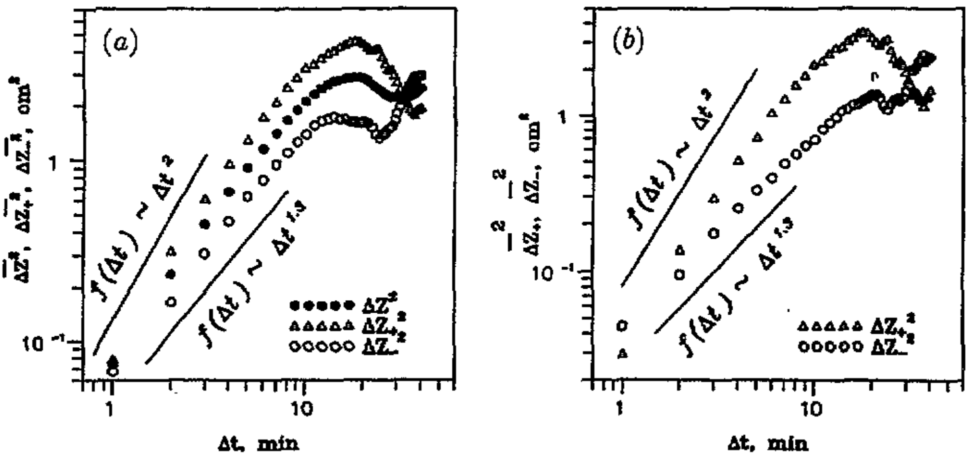

Figure 5. Characteristic graphs of functions $\overline{\Delta Z^2}$, $\overline{\Delta Z_+^2}$, $\overline{\Delta Z_-^2}$ (*a*) and $\overline{\Delta Z_+}^2$, $\overline{\Delta Z_-}^2$ (*b*) for time profiles $Z(t)$ of bottom sand waves (laboratory experiments).

The coincidence of scaling exponents for positive increments in functions $Z(x)$ and $Z(t)$ show the movement of forward faces of sand waves practically without distortions (or that the velocities of their movement are independent of scale: $\overline{\Delta Z_+} \sim \Delta x \sim \Delta t$). However, the difference in scaling exponents for negative increments in functions $Z(x)$ and $Z(t)$ provides grounds for a preliminary conclusion concerning the existence of a peculiar dispersion for sand waves back faces, i.e., the dependence of the velocity of their movement U_- upon scale: $\overline{\Delta Z_-} \sim \Delta t^{-0.65} \sim [\Delta x/U_-(\Delta x)]^{0.65} \sim \Delta x^{0.80}$. This is indicative of the decrease of longitudinal movement velocity U_- of sand waves back faces with the increase of their scale: $U_- \sim \Delta x^{-0.15}$. Thus the utilization of the proposed method has revealed several important peculiarities of sand wave dynamics which are inaccessible to other methods [10].

(C). To analyse the structure of velocity fluctuation in rivers, within the framework of the proposed approach, we have used the measurements of longitudinal velocities on the Chiugur River (Moldova). The applied equipment and methods are described in [10]. The characteristic graphs of functions (1), (3)–(6) for the longitudinal component of velocity vector are given in figure 6. In the range of small time scales of all graphs, the scaling behaviour of structural functions occurs with $H = 0.32$–0.34, which according to equation (2) agrees with Kolmogorov's well known law '2/3' [7]. In all cases the positive increments are stronger than the negative ones, which indicates a faster increase of longitudinal velocities in comparison with their subsidence. In other words, the short intervals of velocity increase alternately with longer ones of gradual velocity decrease. This result is also in good agreement with the conclusions of the local-isotropic turbulence theory [7]. To this it should be added that the size of fluctuations in positive acceleration exceeds the size of fluctuations in deceleration ($\overline{\Delta U_+'^2} > \overline{\Delta U_-'^2}$). This result needs further checking and analysis.

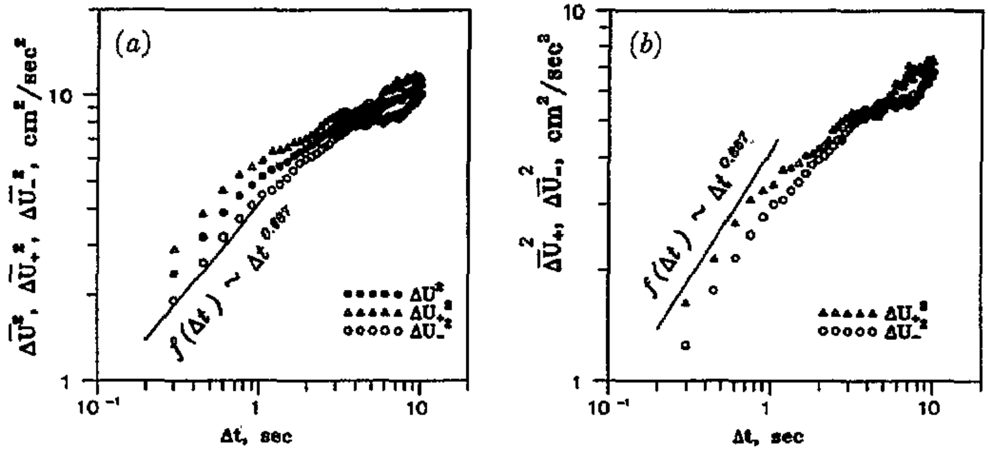

Figure 6. Characteristic graphs of functions $\overline{\Delta U^2}$, $\overline{\Delta U_+^2}$, $\overline{\Delta U_-^2}$ (*a*) and $\overline{\Delta U_+}^2$, $\overline{\Delta U_-}^2$ (*b*) for longitudinal velocities $U(t)$ (measurements in the Chiugur River, Moldova).

In this short communication we have presented the idea of a new method for revealing and analysing scaling properties, which, to a certain degree, adds to the information about natural objects that is obtained on the basis of usual structural functions. By illustrating this idea with different examples we have not set ourselves the task of their detailed physical analysis. We have only demonstrated the possibility of the new approach in obtaining additional information which was inaccessible with traditional approaches such as Hurst's, Richardson's and box-counting methods.

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